# Dynamics and geometry near resonant bifurcations

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# Chapter 2

# Recognition problem: Map case

## 2.1 Introduction

**Resonance.** Here we focus on resonance sets and their boundaries near non-degenerate and mildly degenerate Hopf-Neĭmarck-Sacker bifurcations in families of diffeomorphisms. Such bifurcations occur if one of the maps in such a family, which we call the *central singularity*[1], has a fixed point at which the eigenvalues are on the unit circle. The present study zooms in on the case where this occurs at a $q$-th root of unity. Recall that resonance sets are regions in parameter space corresponding to the occurrence of $q$-periodic orbits near the fixed point of the central singularity, and the boundaries of these sets correspond to the appearance or disappearance of such periodic orbits, typically through a saddle-node bifurcation. In the non-degenerate case of weak resonance, i.e., when $q \geq 5$, a pair of $q$-periodic orbits appears or disappears upon passage of the boundary of an Arnol'd resonance tongue. Such bifurcations occur in generic 2-parameter families. If $q \geq 7$ we encounter a more degenerate situation, which gives rise to the appearance or disappearance of up to four $q$-periodic orbits near the HNS bifurcation. These bifurcations occur in generic 4-parameter families.

**Main results and outline of the chapter.** In [BGV03] normal forms for such families of maps are obtained by applying $\mathbb{Z}_q$-equivariant contact-equivalence singularity theory. These normal forms, depending on two and four parameters, respectively, determine the geometry of the resonance sets in generic families. In this chapter we solve the *recognition problem* for families of maps exhibiting a weak resonance, i.e., we derive a finite set of conditions distinguishing families of maps with diffeomorphically different resonance sets. As usual, these conditions are polynomial equalities and inequalities in finite order derivatives of the map at the bifurcation point. These conditions are obtained via *Lyapunov-Schmidt reduction*, a procedure mapping a family of maps near $p : q$-resonance to a family of $\mathbb{Z}_q$-equivariant

[1] We use the term central singularity for such a map, since it corresponds to a parameter value, which is the (local) central singularity in parameter space.

functions, such that the zeros of the latter family correspond to fixed points or periodic orbits of the former family. The discriminant set of the reduced family is a stratified subset of parameter space, separating open regions in parameter space corresponding to different numbers of zeros. Therefore, this set also separates open regions corresponding to different numbers of $q$-periodic orbits of the family of maps, so it coincides with the boundaries of the resonance set.

Our main contribution is two-fold:

1. The derivation of an algorithm that computes explicit expressions for the Lyapunov-Schmidt reduction of a given resonant family of maps;
2. Using these expressions to solve the recognition problem, based on $\mathbb{Z}_q$-equivariant contact-equivalence singularity theory along the lines of [GSS85, GS88].

The main results are presented in Section 2.2, in which we also present several case studies illustrating our approach. Section 2.3 contains the details of the Lyapunov-Schmidt reduction algorithm, and the output in a characteristic case. Finally, in Section 2.4 we apply $\mathbb{Z}_q$-equivariant contact-equivalence singularity theory to derive conditions characterizing a class of generic and mildly degenerate families of planar diffeomorphisms, and, therefore, solving the recognition problem for such families. In Chapter 3 we analyze the complete geometry of the resonance set of this mildly degenerate family via 2- and 3-dimensional tomograms, i.e., using 2- and 3-dimensional cross-sections of 4-dimensional parameter space.

**Related work.** In many applications, the family of planar diffeomorphisms is obtained from a family of vector fields by taking a suitable Poincaré map corresponding to a section transverse to a periodic orbit. The eigenvalues of the derivative of the Poincaré map then are the Floquet exponents of the periodic orbit. The corresponding bifurcation of the periodic orbit, in particular related to subharmonic periodic solutions and invariant tori, usually is referred to as a Hopf-Neĭmark-Sacker bifurcation, compare with [Kuz95]. This approach also works for non-autonomous systems of differential equations, depending periodically on time.

The kind of resonance scenario discussed here, occurs in many other situations as well. A toy model for the array of tongues of the Hopf-Neĭmark-Sacker bifurcation is formed by the Arnol'd family of circle maps [Arn82], given by $x \mapsto x + 2\pi\alpha + \beta \sin x$. Here in the $(\alpha, \beta)$-plane tongues appear with their tips in $(\alpha, \beta) = (\frac{p}{q}, 0)$ and stretching out into the regions $\beta \neq 0$, see Figure 2.1. Also compare with [Dev89, BGV03, BGV07].

The research program of Peckham *et al.* reflected in [PK02, MP96, MP95, PFK95, PK91] views the boundary of resonance sets as projections on the parameter plane of (saddle-node) bifurcation sets in the product of parameter and phase space. This approach has the same spirit as ours and many interesting geometric properties of 'resonance sets' are discovered and explained in this way. We note that the earlier result [PK91] on higher order degeneracies in a period-doubling uses $\mathbb{Z}_2$-equivariant singularity theory.

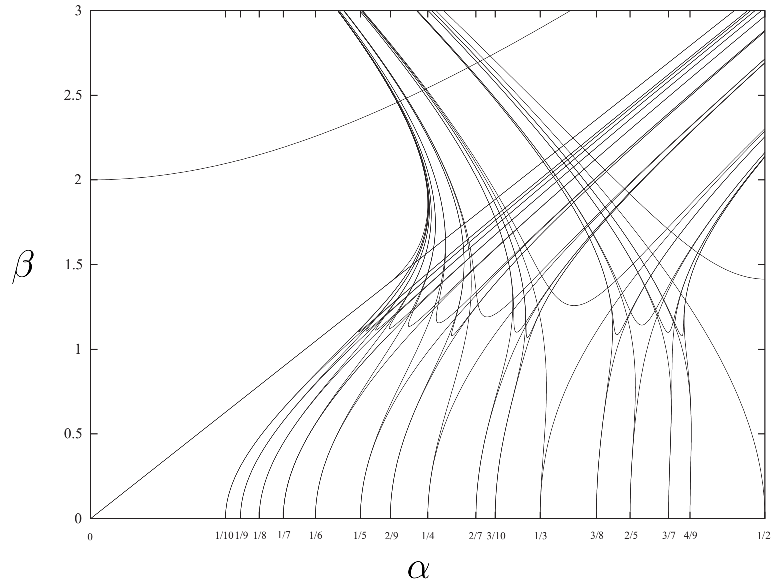


Figure 2.1: Resonance tongues in the Arnol'd family [BST98].

Particularly, we like to mention the results of [PK02] concerning a class of oscillators with doubly periodic forcing. It turns out that these systems can have coexistence of periodic attractors (of the same period), giving rise to 'secondary' saddle-node lines, sometimes enclosing a flame-like shape.

In [Che88, Che85a, Che85b] Chenciner considered a 2-parameter unfolding of a degenerate Hopf bifurcation. Strong resonances to some finite order are excluded in the 'rotation number' $\omega_0$ at the central fixed point. In [Che85b] for sequences of 'good' rationals $p_n/q_n$ tending to $\omega_0$, corresponding fixed points are studied with the help of $\mathbb{Z}_{q_n}$-equivariant normal form theory. For a further discussion of the codimension $k$ Hopf bifurcation see [BR01].

For background regarding weak and strong resonances we refer to Takens [Tak74a], Newhouse *et al.* [NPT83], Arnol'd [Arn82], Krauskopf [Kra94] and Broer, Golubitsky and Vegter [BGV03, BGV07] and [Hum79]. Resonance is also studied in Hamiltonian or reversible settings, etc., compare with Broer *et al.* [BV92,BCKV93,BCKV95,BHLV03,BHLV98b,BL95, BS00, BS98, BV92, BHLV98a], Vanderbauwhede [Van92] or [LM09].

For open problems concerning analytic families of maps on the complex plane, see [Ily08].

## 2.2 Universal deformations and recognition conditions

The main results of this chapter concern a Lyapunov-Schmidt reduction for the problem of detecting $q$-periodic orbits and the precise formulation of the recognition conditions that determine the classification of bifurcation diagrams. We also consider a few case studies.

### 2.2.1 Lyapunov-Schmidt reduction

We consider a family of local diffeomorphisms on the plane, depending on a multi-parameter, such that for a certain parameter value the diffeomorphism has a fixed point with a pair of primitive $q$-th roots of unity as eigenvalues. Such a family may also originate from reduction of a higher dimensional family to a 2-dimensional center manifold. For convenience we identify the plane with $\mathbb{C}$. If the eigenvalues of the linear part at the fixed point are of the form $\omega$ and $\overline{\omega}$, with $\omega = e^{2\pi i p/q}$, the family of diffeomorphisms can be brought into the form

$$P_\mu(z) = (\omega + a_{10}(\mu))\, z + \sum_{2\le i+j<q} a_{ij}(\mu)\, z^i\, \overline{z}^j + O(|z|^q), \tag{2.1}$$

*after* translating the fixed point of $P_\mu$ to the origin and *after* bringing the linear part of $P_\mu$ at this fixed point into Jordan normal form. We assume that $\mu \in \mathbb{R}^n$ and that $\omega = e^{2\pi i p/q}$, with $0 < |p| < q$ coprime. The coefficients $a_{ij}$ are complex valued functions of $\mu$. We note that $a_{10}$ measures detuning and change in stability from the resonant linear map $z \mapsto \omega z$. Lyapunov-Schmidt reduction of the family $P_\mu$ yields a smooth, non-analytic, $\mathbb{Z}_q$-equivariant family $G_\mu : \mathbb{C} \to \mathbb{C}$ of the form

$$G_\mu(z) = z\, B_\mu(u) + C_\mu\, \overline{z}^{q-1} + O(|z|^q), \tag{2.2}$$

where $B_\mu$ is a polynomial in $u = |z|^2$ of degree less than $(q-1)/2$, with $B_\mu(0) = a_{10}(\mu)$. (We discuss Lyapunov-Schmidt reduction in more detail in Section 2.3.) Since the $q$-periodic points of the family of maps $P_\mu$ correspond to the zeros of the reduced family of functions $G_\mu$, the boundary of the resonance sets of the family of maps corresponds to the *discriminant set* $\Sigma(G_\mu)$ of the family $G_\mu$, i.e., the set of parameters for which there is a $z \in \mathbb{C}$ such that

$$G_\mu(z) = 0 \text{ and } \det(D_z G_\mu(z)) = 0.$$

Therefore, determining an explicit form of the Lyapunov-Schmidt reduced family $G_\mu$ is a key step towards the explicit computation of the resonance sets. This is precisely the first part of the main result in the present chapter, i.e., the computation of the functions $B_\mu$ and $C_\mu$ in (2.2) as a function of the coefficients $a_{ij}$ in (2.1).

**Theorem 2.2.1. (Lyapunov-Schmidt reduction)** *Let $P_\mu : \mathbb{C} \to \mathbb{C}$ be a family of diffeomorphisms of the form*

$$P_\mu(z) = (\omega + a_{10}(\mu))\, z + Q_\mu(z), \tag{2.3}$$

*where $\omega = e^{2\pi i p/q}$, with $p$ and $q$ relatively prime and $q > 1$, $a_{10}(0) = 0$ and*

$$Q_\mu(z) = \sum_{2\le i+j<q} a_{ij}(\mu)\, z^i\, \overline{z}^j + O(|z|^q). \tag{2.4}$$

1. *Lyapunov-Schmidt reduction turns q-periodic orbits of $P_\mu$ into zeros of the family $G_\mu$ of $\mathbb{Z}_q$-equivariant maps, which is of the form*

$$G_\mu(z) = z\, B_\mu(u) + C_\mu\, \overline{z}^{q-1} + O(|z|^q), \tag{2.5}$$

*where $B_\mu$ is a polynomial in $u$ of degree less than $(q-1)/2$ of the following form,*

$$B_\mu(u) = a_{10}(\mu) + b_1(\mu)\, u + b_2(\mu)\, u^2 + O(u^3).$$

*Expressions for $B_\mu$ and $C_\mu$ are computed by the Lyapunov-Schmidt reduction algorithm, presented in Section 2.3. An example of these expressions is given at the end of that section.*

2. *If $Q_\mu(z)$ is $\mathbb{Z}_q$-equivariant in the sense that $Q(\omega\, z) = \omega\, Q(z)$, then Lyapunov-Schmidt reduction yields a $\mathbb{Z}_q$-equivariant family $G_\mu$ of the form*

$$G_\mu(z) = a_{10}(\mu)\, z + Q_\mu(z).$$

Section 2.3 contains the proof of this theorem. In fact, we will present an algorithm performing Lyapunov-Schmidt reduction for families of the form (2.1).

**Remarks 2.2.1.**

1. With regard to the second part of the latter theorem note that equivariance of $Q_\mu$, at least up to order $O(|z|^q)$, is equivalent to all coefficients $a_{ij}$ being zero, except possibly for $j = i - 1$ and $(i, j) = (0, q - 1)$.

2. Lyapunov-Schmidt reduction ensures that the $\mathbb{Z}_q$-symmetry of the $q$-periodic of $P_\mu$ becomes a full $\mathbb{Z}_q$-symmetry of the family $G_\mu$.

### 2.2.2 The recognition problem

**$\mathbb{Z}_q$-equivariant contact-equivalence.** We use equivariant singularity theory to study the zero set of $\mathbb{Z}_q$-equivariant families $G_\mu$ of the form (2.2) obtained by Lyapunov-Schmidt reduction. In [BGV03] equivariant singularity theory is applied to obtain normal forms for the simplest $\mathbb{Z}_q$-equivariant germs obtained from Lyapunov-Schmidt reduction. Here we focus on the *recognition problem* for families of germs, i.e., we derive conditions guaranteeing that a given family unfolds one of these simple germs, and does so 'generically'. To this end we first recall some notions and properties of equivariant singularity theory that we apply, see also [GSS85, Chapter III].

We derive equivariant conditions characterizing the orbits of such families under the group of $\mathbb{Z}_q$-equivariant *contact-equivalence transformations*. This group consists of all pairs $(S, Z)$, where $Z : \mathbb{C} \to \mathbb{C}$ is a $\mathbb{Z}_q$-equivariant (local) change of coordinates, i.e.,

$$Z(\omega z) = \omega Z(z) \tag{2.6}$$

for all $z$, with $\omega = e^{2\pi i p/q}$, and the smooth map germ $S(z) : \mathbb{C} \to \mathbb{C}\backslash\{0\}$, satisfies

$$S(\omega z)\,\omega = \omega\, S(z) \tag{2.7}$$

for all $z$. The product of the two group elements $(S, Z)$ and $(T, Y)$ is given by $(S \cdot T, Z \circ Y)$. Moreover, this group acts on the ring of functions (or, rather, germs) as follows, $(S, Z)$ maps the germ $g$ onto the germ $h$ defined by

$$h(z) = S(z)\, g(Z(z)). \tag{2.8}$$

In this case $g$ and $h$ are $\mathbb{Z}_q$*-contact-equivalent*. Note that $Z$ maps the zero set of $h$ to the zero set of $g$, an important feature in our approach of resonances.

A $n$-parameter *unfolding* of a germ $g : \mathbb{C} \to \mathbb{C}$ is a family $G_\mu : \mathbb{C} \to \mathbb{C}$, such that $G_0(z) = g(z)$. We call the germ $g$ the *central singularity* of $G_\mu$. Such an unfolding is $\mathbb{Z}_q$-equivariant if every germ in the family $G_\mu$ is $\mathbb{Z}_q$-equivariant. Two $n$-parameter $\mathbb{Z}_q$-equivariant unfoldings $G_\mu$ and $H_\mu$ of the germ $g$ are called $\mathbb{Z}_q$-contact-equivalent if there is a smooth $n$-parameter family $(S_\mu, Z_\mu)$ of $\mathbb{Z}_q$-equivariant contact-equivalence transformations mapping $G_\mu$ onto $H_\mu$, i.e.:

$$H_\mu(z) = S_\mu(z)\, G_\mu(Z_\mu(z)),$$

such that $S_0(z) = 1$ and $Z_0(z) = z$.

Moreover, an $m$-parameter $\mathbb{Z}_q$-equivariant unfolding $H_\nu$ of a $\mathbb{Z}_q$-equivariant germ $g$ is *induced* by a $n$-parameter $\mathbb{Z}_q$-equivariant unfolding $G_\mu$ of $g$ if there is a smooth map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$, with $\varphi(0) = 0$, and a $\mathbb{Z}_q$-contact-equivalence transformation $(S_{\varphi(\nu)}, Z_{\varphi(\nu)})$, such that

$$H_\nu(z) = S_{\varphi(\nu)}(z)\, G_{\varphi(\nu)}(Z_{\varphi(\nu)}(z)).$$

In this case, $Z_{\varphi(\nu)}$ maps the (singular) zero set of $H_\nu$ onto the (singular) zero set of $G_{\varphi(\nu)}$ and so the discriminant set $\Sigma(H_\nu)$ is obtained by pulling back $\Sigma(G_\mu)$ through the reparametrization $\varphi$. Given the family $G_\mu$, our strategy will be to compute a *normal form*, i.e., a 'simple family' $\mathbb{Z}_q$-contact-equivalent to $G_\mu$. Although not unique, such a normal form can usually be chosen to be a low-degree polynomial family $H_\nu$, the discriminant set $\Sigma(H_\nu)$ of which can be determined by a straightforward calculation. This notion of 'simple family' is made precise by considering *versal unfoldings*, i.e., unfoldings induced by any other unfolding of the same germ. An unfolding is called *universal* if it is versal with a minimal number of parameters. This minimal number of parameters is the *codimension* of the germ it unfolds. Since resonance sets correspond to discriminant sets of Lyapunov-Schmidt reduced functions, it is obvious why it is important to study (uni)versal unfoldings of these reduced functions.

**Versal unfoldings of $\mathbb{Z}_q$-equivariant functions.** The second part of the main result of the current chapter consists of precise conditions characterizing $\mathbb{Z}_q$-equivariant families with non-degenerate or mildly degenerate central singularities. These conditions solve the *recognition problem* for such families.

**Theorem 2.2.2. (Recognition conditions)** *Let $G_\mu : \mathbb{C} \to \mathbb{C}$ with the multi-parameter $\mu \in \mathbb{R}^n$ be a $\mathbb{Z}_q$-equivariant family of the form*

$$G_\mu(z) = z\, K_\mu(u) + L_\mu\, \overline{z}^{q-1} + O(|z|^q),$$

*with $u = |z|^2$, $K_0(0) = 0$ and $L_0 \neq 0$. Let $g = G_0$ be the central singularity of the family $G_\mu$.*

1. *If $D_u K_0(0) \neq 0$, $q \geq 5$, then $G_\mu$ is a versal unfolding of $g$ if*

$$\mu \mapsto (\mathrm{Re}(K_\mu(0)), \mathrm{Im}(K_\mu(0))) \tag{2.9}$$

   *is a submersion at $\mu = 0$. $G_\mu$ is a universal unfolding if $n = 2$.*

2. *If $D_u K_0(0) = 0$, $D_u^2 K_0(0) \neq 0$ and $q \geq 7$, then $G_\mu$ is a versal unfolding of $g$ if*

$$\mu \mapsto (\mathrm{Re}(K_\mu(0)), \mathrm{Im}(K_\mu(0)), \mathrm{Re}(D_u K_\mu(0)), \mathrm{Im}(D_u K_\mu(0))) \tag{2.10}$$

   *is a submersion at $\mu = 0$. $G_\mu$ is a universal unfolding if $n = 4$.*

**Remark 2.2.1.** Gradual violation of the degeneracy conditions like $L_0 \neq 0$ gives rise to a familiar endless sequence of bifurcations of ever higher codimension.

We prove this result in Section 2.4, which also contains the corresponding results for the strong resonances $q = 3$ and $q = 4$.

Theorem 2.2.2 allows us to derive explicit expressions for universal unfoldings of certain $\mathbb{Z}_q$-equivariant germs.

**Corollary 2.2.1.** *Let $g$ be a $\mathbb{Z}_q$-equivariant germ of the form*

$$g(z) = z\, k(u) + \ell\, \overline{z}^{q-1} + O(|z|^q),$$

*with $k(0) = 0$ and $\ell \neq 0$. A universal unfolding of $g$ is of the form*

$$G_\sigma(z) = g(z) + \sigma\, z, \qquad \text{if } D_u k(0) \neq 0; \tag{2.11}$$

$$G_{\sigma,\tau}(z) = g(z) + \sigma\, z + \tau\, z\, |z|^2, \quad \text{if } D_u k(0) = 0 \text{ and } D_u^2 k(0) \neq 0, \tag{2.12}$$

*where the parameters $\sigma = \sigma_1 + i\sigma_2$ and $\tau = \tau_1 + i\tau_2$ are complex.*

**Remark 2.2.2.** In [BGV03] we also derived normal forms $h(z)$ for the central singularity $g$ in these cases:

$$h(z) = z\, |z|^2 + \overline{z}^{q-1}, \text{ in case (2.11);}$$
$$h(z) = z\, |z|^4 + \overline{z}^{q-1}, \text{ in case (2.12).}$$

Germs of this kind, and their universal unfoldings, will show up in the case studies presented in Section 2.2.3.

**The recognition problem for families of planar diffeomorphisms.** We now apply Theorem 2.2.2 on versality of general unfoldings of $\mathbb{Z}_q$-equivariant germs to obtain conditions solving the *recognition problem for families of planar diffeomorphisms* of the form $P_\mu(z) = (\omega + a_{10}(\mu))\, z + Q_\mu(z)$, where $Q_\mu$ is of the form (2.4), with $q$-periodic orbits bifurcating from a fixed point $z = 0$ at $\mu = 0$. The reduced family $G_\mu(z)$ is of the form (2.5), i.e.,

$$G_\mu(z) = z B_\mu(u) + C_\mu\, \overline{z}^{q-1} + O(|z|^q),$$

where $u = |z|^2$ and

$$B_\mu(u) = a_{10}(\mu) + b_1(\mu)\, u + b_2(\mu)\, u^2 + O(u^3).$$

**Corollary 2.2.2.** *Assume that the coefficient $C_0$ of the term $\overline{z}^{q-1}$ in* (2.5) *is nonzero.*

1. *If $a_{10}(0) = 0$, $b_1(0) \neq 0$ and $q \geq 5$, then the family $G_\mu(z)$ is a versal unfolding of the germ $G_0(z)$ if*
$$\mu \mapsto (\mathrm{Re}(a_{10}(0)), \mathrm{Im}(a_{10}(0))) \tag{2.13}$$
*is a submersion at $\mu = 0$.*

2. *If $a_{10}(0) = b_1(0) = 0$, $b_2(0) \neq 0$ and $q \geq 7$, then the family $G_\mu(z)$ is a versal unfolding of the germ $G_0(z)$ if*
$$\mu \mapsto (\mathrm{Re}(a_{10}(0)), \mathrm{Im}(a_{10}(0)), \mathrm{Re}(b_1(0)), \mathrm{Im}(b_1(0))) \tag{2.14}$$
*is a submersion at $\mu = 0$.*

*Proof.* The condition $C_0 \neq 0$ is equivalent to assuming that $L_0 \neq 0$ in Theorem 2.2.2.

1. First consider the non-degenerate case, corresponding to $b_1(0) \neq 0$ and $q \geq 5$. As $B_\mu$ corresponds to $K_\mu$ in Theorem 2.2.2, it follows that $B_0(0) = a_{10}(0)$ and $D_u B_0(0) = b_1(0)$ immediately imply Corollary 2.2.2.

2. In the mildly degenerate case, replacing $K_\mu$ in Theorem 2.2.2 with $B_\mu$, where $B_0(0) = a_{10}(0)$, $D_u B_0(0) = b_1(0)$, $D_u^2 B_0(0) = b_2(0)$ and taking $q \geq 7$, immediately implies Corollary 2.2.2.

□

### 2.2.3 Case studies

To illustrate our main result we present some examples in which we determine resonance sets of planar families of diffeomorphisms a fixed point of which undergoes a Hopf-Neĭmark-Sacker bifurcation with its characteristic array of resonance sets organizing the alternation of periodic and quasi-periodic dynamics. We zoom in on the shape of one such set, as a subset of the resonance bifurcation diagram, briefly reviewing the classical non-degenerate case, but then turning to a next case of degeneracy. For further examples of such bifurcations we refer to [BSV02, BSV08, BSV02, Kuz95].

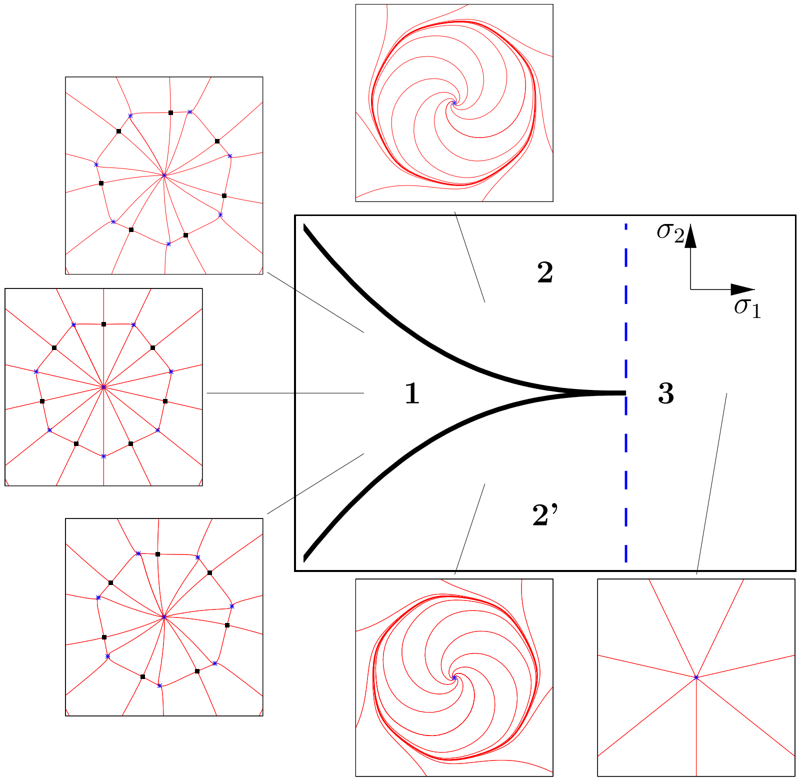


Figure 2.2: The non-degenerate resonance set of $P_\sigma$ given in (2.15) in the $\sigma$-plane for $q = 7$ and $b = d = \omega$ (central picture), with phase portraits for various values of the parameter $\sigma$. For $\sigma$ in region 1 the map $P_\sigma$ has a fixed point surrounded by two period-7 orbits. Curves between the periodic points are their stable or unstable manifolds. If $\sigma$ is changed such that it crosses the boundary of the resonance set, the two period-7 orbits disappear in a saddle-node bifurcation. Regions 2 and 2' correspond to phase portraits with a single fixed point enclosed in an invariant circle. The complete bifurcation diagram consists of the boundary of the resonance set and a Hopf-line given by $\sigma_1 = 0$. When the parameter crosses this line into region 2 or region 2' the invariant circle disappears.

**Generic equivariant families.** The first example is a non-degenerate planar family of diffeomorphisms near a $p : q$-resonance with a $\mathbb{Z}_q$-equivariant $(q-1)$-jet, i.e., a family $P_\sigma(z)$ given by

$$P_\sigma(z) = (\omega + \sigma)\, z + b\, z\, |z|^2 + d\bar{z}^{q-1} + O(|z|^q), \tag{2.15}$$

where $\sigma \in \mathbb{C}$ is a complex parameter ranging over a neighborhood of $0 \in \mathbb{C}$ and where $b$ and $d$ are non-zero complex constants, and $\omega = e^{2\pi i p/q}$, with $p$ and $q \geq 5$ coprime. Theorem 2.2.1, and the subsequent observations, show that Lyapunov-Schmidt reduction yields the reduced family

$$G_\sigma(z) = \sigma\, z + b\, z\, |z|^2 + d\bar{z}^{q-1} + O(|z|^q).$$

A straightforward derivation along the lines of Appendix A.1.1 and A.1.2 shows that the discriminant set of this family, and, hence, the resonance set boundary of the family (2.15), is a familiar $(q-2)/2$-cusp in the $\sigma$-plane. The resonance set, and some characteristic phase portraits, are plotted in Figure 2.2 for $q = 7$. The complete bifurcation set consists of the boundary of the resonance set and a Hopf-line, corresponding to the appearance or disappearance of an invariant circle. This Hopf-line is determined by other (standard) methods from bifurcation theory. Further details are provided in the caption of Figure 2.2.

**A higher dimensional example: Reduction to a center manifold.** Our second example is a family of diffeomorphisms in $\mathbb{R}^3$, with two conjugate 5-th roots of unity as eigenvalues, and a third real eigenvalue off the unit circle. We determine for which values of the parameter resonances of order 5 may occur, and show how the resonance sets can be determined by restricting to a 2-dimensional center manifold.

The example is inspired by a related system studied in [BSV08], where the resonance sets are determined by first computing a Poincaré-Takens normal form for the family of maps. Then the fixed points corresponding to this type of resonance are located, and finally an expression for the boundary of the resonance set is obtained by looking for saddle-node bifurcations of the points of period 5.

We follow an alternative approach by applying the methods developed in this chapter to determine the resonance sets via Lyapunov-Schmidt reduction. The family of maps $F_\mu : \mathbb{R}\times\mathbb{C} \to \mathbb{R}\times\mathbb{C}$ is given by

$$F_\mu(x,z) = (F_1(x,z), F_2(x,z;\mu)),$$

with

$$\begin{aligned} F_1(x,z) &= x + b^2 - x^2 - |z|^2, \\ F_2(x,z;\mu) &= z\,(\omega + \mu - a\,x - x^2) + d\,\overline{z}^4. \end{aligned}$$

Here $\mu$ is a complex parameter, to be specified later on, $b$ is a non-zero real constant, and $a$ and $d$ are complex constants. Furthermore, $\omega = 2\pi i p/5$, with $0 < p < 5$. The map has fixed points at $(\pm b, 0)$, with one real eigenvalue equal to $1-2b$, which is off the unit circle since $b \neq 0$. We are interested in the occurrence of $p:5$ resonances at these fixed points. To this end we focus on the fixed point $(b,0)$, and impose the condition that the linear part of $F_\mu$ of this fixed point also has two eigenvalues $\omega$ and $\overline{\omega}$ lying on the unit circle. A short calculation shows that this situation occurs for $\mu = a\,b + b^2$. For convenience we introduce the parameter $\sigma$, defined by $\sigma = \mu - a\,b - b^2$, ranging over a neighborhood of $0 \in \mathbb{C}$. At the fixed point $(b,0)$ the family $F_\sigma$ has a center manifold of the form

$$x = \varphi_\sigma(z) = b + c_\sigma\,|z|^2 + O(|z|^3).$$

We determine the unknown real coefficient $c_\sigma$ from the invariance condition

$$F_1(\varphi_\sigma(z), z) = \varphi_\sigma(F_2(\varphi_\sigma(z), z; \sigma)).$$

A straightforward calculation shows that

$$c_\sigma = \frac{1}{1 - 2b - |\omega + \sigma|^2} = -\frac{1}{2b} + O(|\sigma|).$$

Restricted to the invariant manifold the map $F_\sigma$ is of the form

$$z \mapsto F_2(\varphi_\sigma(z), z; \sigma) = (\omega + \sigma)\, z + (\frac{a}{2b} - 1 + O(|\varrho|))\, z\, |z|^2 + d\, \overline{z}^4 + O(|z|^5).$$

Note that this family is of the form (2.15), provided $a \neq 2b$, so the analysis of the first case applies to this system as well. It leads to a standard cusp shaped resonance set like the one depicted in Figure 2.2.

**Mildly degenerate equivariant families.** As in the first case study, our third example is a planar family $P_{\sigma,\tau}(z)$ in normal form, i.e.,

$$P_{\sigma,\tau}(z) = (\omega + \sigma)\, z + \tau\, z\, |z|^2 + c\, z\, |z|^4 + d\bar{z}^{q-1} + O(|z|^q), \tag{2.16}$$

where $\sigma = \sigma_1 + i\sigma_2$ and $\tau = \tau_1 + i\tau_2$ are complex parameters ranging over a small neighborhood of $0 \in \mathbb{C}$, and $c$ and $d$ are nonzero complex constants. Note that this family is slightly more degenerate than (2.15), since also the coefficient $\tau$ of the third order term is a small parameter. Furthermore, we require $q \geq 7$, cf. Theorem 2.2.2. Lyapunov-Schmidt reduction yields the reduced family

$$G_{\sigma,\tau}(z) = \sigma\, z + \tau\, z\, |z|^2 + c\, z\, |z|^4 + d\bar{z}^{q-1} + O(|z|^q). \tag{2.17}$$

In Chapter 3 we present a full description of the resonance set, i.e., of the discriminant set of the family $G_{\sigma,\tau}$, which is now an algebraic hypersurface in 4-dimensional parameter space. In Figure 2.3 we depict a 2-dimensional intersection of this parameter space for $q = 7$, together with some phase portraits. Note that, again, the resonance sets do not represent all local bifurcations of the family (2.16). In Chapter 3 we provide a more detailed description of the bifurcation set.

**Remark 2.2.3.** The case studies presented in this section all start from a rather simple expression (normal form) of the family of diffeomorphisms. The Lyapunov-Schmidt algorithm, to be presented in Section 2.2.1, reduces arbitrarily complicated expressions. See the example at the end of Section 2.2.1.

## 2.3 An algorithm for Lyapunov-Schmidt reduction

The algorithm for Lyapunov-Schmidt reduction performs three steps (A `C++` implementation of this algorithm, and some sample output, can be obtained from [Hol]). First it transforms the problem of finding $q$-periodic orbits in a family of maps $P_\mu : \mathbb{C} \to \mathbb{C}$, bifurcating from a

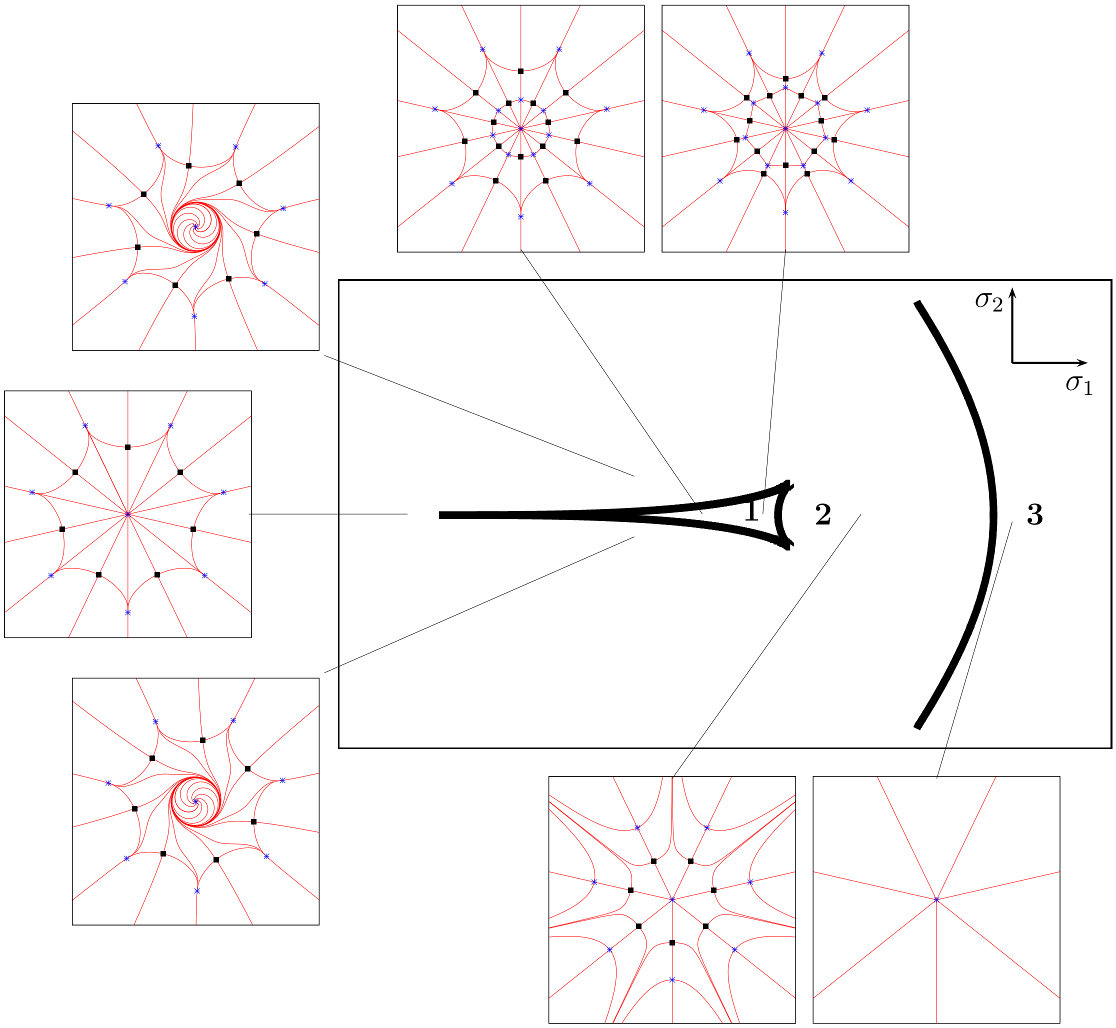


Figure 2.3: The resonance set for the mildly degenerate family (2.16) in a 2-dimensional cross-section, with $\tau$ fixed, of $(\sigma, \tau)$-parameter space for $q = 7$ (central picture), with phase portraits for various values of the parameter $\sigma$ in the complement of the resonance set (regions 1, 2 and 3). The origin $\sigma = 0$ is at the tip of the triangular region 1. Regions 1, 2 and 3 correspond to the occurrence of 4, 2, and 0 period-7 orbits. The 4 period-7 orbits lie on two disjoint invariant circles. If $\mathrm{Im}(\sigma)$ varies such that $\sigma$ crosses the boundary of region 1, two period-7 orbits on the inner circle disappear in a saddle-node bifurcation. On the other hand, if $\mathrm{Re}(\sigma)$ increases such that $\sigma$ crosses boundary between region 2 and region 3, a periodic orbit of the inner circle and one of the outer circle disappear in a saddle-node bifurcation, destroying both invariant circles. Crossing the boundary between region 2 and 3 yields another saddle-node bifurcation destroying the remaining periodic orbits.

fixed point, into the problem of finding zeros of an associated map $\widehat{P}_\mu : \mathbb{C}^q \to \mathbb{C}^q$. Second, under certain conditions on the linear part of $P_\mu$ at the fixed point, the problem is reduced to finding the zeros of a reduced function $G_\mu : \mathbb{C} \to \mathbb{C}$. Third, it determines the function $G_\mu$ to any desired finite order by successive approximation of the solution of an implicit equation.

**Detecting periodic points.** The first step of the algorithm is the construction of the map $\widehat{P}_\mu$, the zeros of which correspond to the $q$-periodic orbits of $P_\mu : V \to V$, where $V$ is some finite dimensional vector space. In our setting $V = \mathbb{C}$ and $P_\mu$ is a family of maps given by (2.1). To this end, let $x_1, \ldots, x_q$ be a $q$-periodic orbit of $P_\mu$. This orbit is the zero of the map $\widehat{P}_\mu : V^q \to V^q$, defined by

$$\widehat{P}_\mu(x_1, \ldots, x_q) = (P_\mu(x_1) - x_2, \ldots, P_\mu(x_{q-1}) - x_q, P_\mu(x_q) - x_1). \tag{2.18}$$

We assume that the periodic orbit bifurcates from a fixed point at the origin, implying $\widehat{P}_\mu(0) = 0$. Moreover, this family is $\mathbb{Z}_q$-equivariant, because it commutes with $\xi : V^q \to V^q$, defined by

$$\xi(x_1, \ldots, x_q) = (x_2, \ldots, x_q, x_1).$$

Clearly, $\xi$ generates an action of $\mathbb{Z}_q$ on $V^q$. The search for $q$-periodic orbits of $P_\mu$ is transformed into computing the zeros of the $\mathbb{Z}_q$-equivariant map $\widehat{P}_\mu$ locally near $0 \in V^q$.

**An algorithm for Lyapunov-Schmidt reduction.** Starting in a more general setting, let $\Phi : W \to W$ be a smooth map on a vector space $V$, having the origin as a fixed point. (In our setting, $\Phi = \widehat{P}$ and $W = V^q$.) Moreover, assume that $D_x\Phi(0)$ is semi-simple, i.e.,

$$W = \ker(D_x\Phi(0)) \oplus \mathrm{im}(D_x\Phi(0)),$$

where $\ker(D_x\Phi(0))$ is the kernel and $\mathrm{im}(D_x\Phi(0))$ the range of $D_x\Phi(0)$. Let $E$ be the projection onto $\mathrm{im}(D_x\Phi(0))$ Then $I - E$ is the projection onto $\ker(D_x\Phi(0))$. Using the variables $s \in \ker(D_x\Phi(0))$ and $t \in \mathrm{im}(D_x\Phi(0))$, it follows that

$$\Phi(s, t) = 0$$

if and only if

$$E\Phi(s, t) = 0 \text{ and } (I - E)\Phi(s, t) = 0. \tag{2.19}$$

The implicit function theorem implies there is a unique map $s \mapsto t(s)$ near $s = 0$, such that $E\Phi(s, t(s)) = 0$. The solution $t(s)$ can be substituted in $(I - E)\Phi(s, t)$ yielding $G(s) \equiv (I - E)\Phi(s, t(s))$. Consequently, the study of zeros of $\Phi : W \to W$ has been reduced to the study of zeros of $G : \ker(D_x\Phi(0)) \to \ker(D_x\Phi(0))$. It should be noted that Lyapunov-Schmidt reduction preserves any equivariance of $\Phi$ when appropriate coordinates are used, cf. [GSS85, Chapter VII.3]. We use the coordinates on $W$ in which $D_x\Phi$ and the symmetry generating map $\xi$ are both diagonal.

To determine the reduced function $G(s) = (I - E)\Phi(s, t(s))$ up to a required order, first $t(s)$ is determined by successive approximation of higher order terms. To this end, rewrite $E\,\Phi(s, t)$ as

$$E\,\Phi(s, t) = (As + Bt) + O_2(s, t) + \cdots + O_k(s, t) + O(|s + t|^{k+1}),$$

where $A = E\,D_x\Phi(0)|_{\ker(D_x\Phi(0))}$, $B = E\,D_x\Phi(0)|_{\mathrm{im}(D_x\Phi(0))}$ and $O_\ell(s,t)$, $1 < \ell \le k$, denotes the homogeneous part of $E\Phi(s,t)$ of total degree $\ell$ in $s$ and $t$. A first order approximation $t^{(1)}(s)$ of $t(s)$ is given by

$$t^{(1)}(s) = -B^{-1}As.$$

Assume the algorithm has computed the approximation $t^{(k-1)}(s)$ of $t(s)$ up to order $O(|s|^k)$, i.e.,

$$t(s) = t^{(k-1)}(s) + O(|s|^k).$$

The approximation of $t(s)$ up to order $O(|s|^{k+1})$ is then given by

$$t^{(k)}(s) = -B^{-1}(As + O_2(s, t^{(k-1)}(s)) + \cdots + O_k(s, t^{(k-1)}(s)))|_k,$$

where $\ldots|_k$ denotes truncation of all terms of order $O(|s|^{k+1})$. Hence, the approximation of the reduced function up to order $O(|s|^{k+1})$ is given by

$$G(s) = (I - E)\Phi(s, t^{(k)}(s)) + O(|s|^{k+1}).$$

**Lyapunov-Schmidt reduction for planar families of diffeomorphisms.** Now we turn to the proof of Theorem 2.2.1, which is based on the application of the algorithm for Lyapunov-Schmidt reduction to the family (2.1), i.e.,

$$P_\mu(z) = (\omega + a_{10}(\mu))\, z + Q_\mu(z),$$

where $a_{10}(0) = 0$ and

$$Q_\mu(z) = \sum_{2 \le i+j < q} a_{ij}(\mu)\, z^i\, \overline{z}^j + O(|z|^q).$$

As before, $\omega = e^{2\pi i p/q}$, with $0 < |p| < q$ coprime integers.

*Proof.* Here we prove Theorem 2.2.1.

1. As for the first part of the proof, observe the map $\widehat{P}_\mu : \mathbb{C}^q \to \mathbb{C}^q$, defined by

$$\widehat{P}_\mu(z_1, \ldots, z_q) = (P_\mu(z_1) - z_2, \ldots, P_\mu(z_{q-1}) - z_q, P_\mu(z_q) - z_1),$$

is of the form

$$\widehat{P}_\mu = \widehat{L}_\mu + \widehat{Q}_\mu,$$

where $\widehat{L}_\mu : \mathbb{C}^q \to \mathbb{C}^q$ is the linear map with matrix

$$\widehat{L}_\mu = \begin{pmatrix} \omega + a_{10}(\mu) & -1 & 0 & \cdots & 0 & 0 \\ 0 & \omega + a_{10}(\mu) & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega + a_{10}(\mu) & -1 \\ -1 & 0 & 0 & \cdots & 0 & \omega + a_{10}(\mu) \end{pmatrix},$$

and $\widehat{Q}$ is the map defined by

$$\widehat{Q}_\mu(z_1, \ldots, z_q) = (\, Q_\mu(z_1), Q_\mu(z_2), \ldots, Q_\mu(z_q)\,).$$

Note that we abuse notation by denoting a linear map and its matrix by the same symbol. The linear transformation $A : \mathbb{C}^q \to \mathbb{C}^q$ bringing $\widehat{L}_0$ into diagonal form is the Vandermonde map, with matrix entries

$$A_{ij} = \omega^{(i-1)j}.$$

Using $\omega^q = 1$, it is easy to prove that its inverse has entries

$$A^{-1}_{ij} = \tfrac{1}{q}\,\omega^{(q-i)(j-1)},$$

and that

$$A^{-1}\,\widehat{L}_\mu\,A = \begin{pmatrix} a_{10}(\mu) & & & \\ & \omega - \omega^2 + a_{10}(\mu) & & \\ & & \ddots & \\ & & & \omega - \omega^q + a_{10}(\mu) \end{pmatrix}. \tag{2.20}$$

Using the linear map with matrix $A$ yields convenient coordinates for solving equations (2.19). To this end, let the map $\Phi_\mu : \mathbb{C}^q \to \mathbb{C}^q$ be defined by

$$\Phi_\mu = A^{-1}\,\widehat{P}_\mu\,A.$$

For $\mu = 0$ the linear part $D_z\Phi_0$ at $0 \in \mathbb{C}^q$ has matrix (2.20), so its kernel is

$$\ker(D_z\Phi_0) = \{(z_1, z_2, \ldots, z_q) \mid z_2 = \cdots = z_q = 0\}.$$

Furthermore, the projection onto $\operatorname{im}(D_z\Phi_0)$ is the map $E : \mathbb{C}^q \to \mathbb{C}^{q-1}$, given by

$$E(z_1, z_2, \ldots, z_q) = (z_2, \ldots, z_q).$$

Using these coordinates, a straightforward application of the successive approximation algorithm for Lyapunov-Schmidt reduction up to order $O(|z_1|^q)$ yields an explicit expression for the function $G_\mu$, introduced in the first part of Theorem 2.2.1.

2. To prove the second part of Theorem 2.2.1, stating that Lyapunov-Schmidt reduction of a family that is already $\mathbb{Z}_q$-equivariant is a trivial operation. We shall prove that

$$\Phi_\mu(z, 0, \ldots, 0) = (a_{10}(\mu)\,z + Q_\mu(z), 0, \ldots, 0). \tag{2.21}$$

Assuming (2.21) holds, the system

$$E\,\Phi_\mu(z_1, z_2, \ldots, z_q) = (0, \ldots, 0) \in \mathbb{C}^{q-1}$$

has solution $z_2 = \cdots = z_q = 0$, locally near $0 \in \mathbb{C}^q$. According to the Implicit Function Theorem this solution is locally unique. Therefore, Lyapunov-Schmidt reduction yields the family $G_\mu : \mathbb{C} \to \mathbb{C}$, given by

$$G_\mu(z) = \Phi_\mu(z, 0, \ldots, 0)_1 = a_{10}(\mu)\, z + Q_\mu(z).$$

So it remains to prove (2.21). It follows from

$$\Phi_\mu = A^{-1}\, \widehat{L}_\mu\, A + A^{-1}\, \widehat{Q}_\mu\, A$$

and (2.20) that we only have to prove

$$A^{-1}\, \widehat{Q}_\mu\, A(z, 0, \ldots, 0) = (Q_\mu(z), 0, \ldots, 0).$$

This identity follows from the following computation:

$$\begin{aligned} A^{-1}\, \widehat{Q}_\mu\, A(z, 0, \ldots, 0) &= A^{-1}\, \widehat{Q}_\mu(z\, A(1, 0, \ldots, 0)) \\ &= A^{-1}\, \widehat{Q}_\mu(z, \omega\, z, \ldots, \omega^{q-1}\, z)) \\ &= A^{-1}(Q_\mu(z), Q_\mu(\omega\, z) \ldots, Q_\mu(\omega^{q-1}\, z)) \\ &= A^{-1}(Q_\mu(z), \omega\, Q_\mu(z), \ldots, \omega^{q-1}\, Q_\mu(z)) \\ &= Q_\mu(z)\, A^{-1}(1, \omega\, \ldots, \omega^{q-1}) \\ &= Q_\mu(z)\, (1, 0, \ldots, 0) \\ &= (Q_\mu(z), 0, \ldots, 0). \end{aligned}$$

This concludes the proof of Theorem 2.2.1. □

**Example output of the Lyapunov-Schmidt algorithm.** To show the kind of output the algorithm generates we consider resonances of order 5, in a family of planar diffeomorphisms. More precisely, consider

$$P_\mu(z) = (\omega + a_{10}(\mu))\, z + \sum_{2 \le i+j < 5} a_{ij}(\mu)\, z^i\, \overline{z}^j + O(|z|^5).$$

where $\omega = e^{2\pi i p/5}$, with $0 < |p| < 4$. Then the reduced function $G_\mu$ is of the form

$$G_\mu(z) = \; B_\mu(u)\; z + \; C_\mu\, \overline{z}^4 + O(|z|^5, a_{10}(\mu)),$$

with $u = |z|^2$, cf. Theorem 2.2.1. The Lyapunov-Schmidt reduction algorithm computes the following expressions for $B_\mu$ and $C_\mu$:

$$B_\mu(u) = a_{10}(\mu) + u(a_{21}(\mu) + b_1\, a_{20}(\mu) a_{11}(\mu) + b_2\, |a_{11}(\mu)|^2 + b_3\, |a_{02}(\mu)|^2),$$

with

$$
\begin{aligned}
b_1 =& \tfrac{1}{5}\left(6\omega^3 + 7\omega^2 + 8\omega + 9\right),\\
b_2 =& -\tfrac{1}{5}\left(\omega^3 + 2\omega^2 + 3\omega - 1\right),\\
b_3 =& \tfrac{2}{5}\left(3\omega^3 + \omega^2 - \omega + 2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
C_\mu = a_{04}(\mu) &+ c_1 a_{11}(\mu) a_{03}(\mu) + c_2 a_{02}(\mu)\overline{a_{30}(\mu)} + c_3 a_{11}(\mu)^2 a_{02}(\mu)\\
&+ c_4 a_{11}(\mu) a_{02}(\mu)\overline{a_{20}(\mu)} + c_5 a_{02}(\mu)^2\overline{a_{11}(\mu)} + c_6 a_{02}(\mu)\overline{a_{20}(\mu)}^2,
\end{aligned}
$$

with

$$
\begin{aligned}
c_1 =& \tfrac{1}{5}\left(4\omega^3 + 3\omega^2 + 2\omega + 1\right),\\
c_2 =& \tfrac{2}{5}\left(3\omega^3 + \omega^2 - \omega + 2\right),\\
c_3 =& \tfrac{2}{5}(\omega - 1)(\omega + 1),\\
c_4 =& \tfrac{1}{5}\left(-4\omega^3 - 3\omega^2 - 2\omega + 4\right),\\
c_5 =& \tfrac{2}{5}\left(\omega^3 + \omega^2 + 3\right),\\
c_6 =& \tfrac{1}{5}\left(-2\omega^3 + 8\omega^2 + 5\omega + 9\right).
\end{aligned}
$$

## 2.4 Recognition problem for planar families

**The (uni)versal unfolding theorem [BGV03, GSS85].** The germs considered in this chapter are elements of $\mathcal{E}(\mathbb{Z}_q)$, the ring of all germs of $\mathbb{Z}_q$-equivariant functions from $\mathbb{C}$ to $\mathbb{C}$ at $0 \in \mathbb{C}$.

The $\mathbb{Z}_q$*-tangent space* $T(g)$ at a germ $g \in \mathcal{E}(\mathbb{Z}_q)$ consists of all germs of the form $\frac{d}{dt} g_t(z)\big|_{t=0}$, where $g_t$ is a 1-parameter family of germs $\mathcal{E}(\mathbb{Z}_q)$ that are $\mathbb{Z}_q$-contact equivalent to $g_0 = g$. In other words, there is a 1-parameter family $(S_t, Z_t)$ of $\mathbb{Z}_q$-equivariant contact transformations, such that

$$g_t(z) = S_t(z)\, g(Z_t(z)),$$

with $S_0(z) = 1$ and $Z_0(z) = z$. Note that $t \mapsto g_t$ is a curve in the orbit of $g$ under the action of the group of $\mathbb{Z}_q$-equivariant contact transformations.

The *(uni)versal unfolding theorem* states that an $n$-parameter unfolding $G_\mu$ of a germ $g$ is *versal* if

$$\mathcal{E}(\mathbb{Z}_q) = T(g) + \mathbb{R}\,\{\left.\frac{\partial G_\mu}{\partial \mu_1}\right|_{\mu=0} + \cdots + \left.\frac{\partial G_\mu}{\partial \mu_n}\right|_{\mu=0}\},$$

and that $G_\mu$ is *universal* if also the number $n$ of parameters in $G_\mu$ equals the codimension of $T(g)$, i.e., the dimension of the real vector space $\mathcal{E}(\mathbb{Z}_q)\,/\,T(g)$.

By the Schwarz finitude theorem [GSS85], every $g \in \mathcal{E}(\mathbb{Z}_q)$ has a unique form:

$$g(z) = K(u,v)\, z + L(u,v)\, \bar{z}^{q-1}, \tag{2.22}$$

where $u = z\bar{z}$ and $v = z^{q-1} + \bar{z}^{q-1}$, and $K$, $L$ are uniquely define complex-valued function germs. Consequently, every $g \in \mathcal{E}(\mathbb{Z}_q)$ can be identified with the pair $(K, L) \in \mathcal{E}^2_{u,v}$, where $\mathcal{E}_{u,v}$ is the ring of smooth complex-valued germs depending on the real variables $u$ and $v$. This identification is used in [BGV03] to determine the $\mathbb{Z}_q$-tangent space of the 'simplest' $\mathbb{Z}_q$-equivariant germs. These results will also be crucial in the proof of Theorem 2.2.2.

*Proof.* Here we prove Theorem 2.2.2. To this end, we consider $\mathbb{Z}_q$-equivariant families $G_\mu$, which are of the form

$$G_\mu(z) = K_\mu(u,v)\, z + L_\mu(u,v)\, \overline{z}^{q-1},$$

for parameter values $\mu$ near $0 \in \mathbb{R}^n$, with $K_0(0,0) = 0$ and $L_0(0,0) \neq 0$, unfolding the central singularity $g$ given by

$$g(z) = K_0(u,v)\, z + L_0(u,v)\, \overline{z}^{q-1}.$$

To prove that $G_\mu$ is a versal unfolding of $g$ under the conditions (2.9) and (2.10), respectively, we use the (uni)versal unfolding theorem. In view of this theorem it is sufficient to show that

$$\mathcal{E}(\mathbb{Z}_q) = T(g) + \mathbb{R}\{ \frac{\partial G_\mu}{\partial \mu_1}\bigg|_{\mu=0}, \ldots, \frac{\partial G_\mu}{\partial \mu_n}\bigg|_{\mu=0} \} \tag{2.23}$$

where $T(g)$ is the $\mathbb{Z}_q$-tangent space of $g$.

Recall that we identify $\mathcal{E}(\mathbb{Z}_q)$ with $\mathcal{E}^2_{u,v}$, and $g \in \mathcal{E}^2_{u,v}(\mathbb{Z}_q)$ with the pair $(K_0, L_0) \in \mathcal{E}^2_{u,v}$. Since $L_0(0,0) \neq 0$, it follows that the second component of $(K_0, L_0)$ generates $\mathcal{E}_{u,v}$. Since $K_0(0,0) = 0$, the first component belongs to the maximal ideal $\mathcal{M}$ of $\mathcal{E}_{u,v}$. So,

$$T(g) \subset \mathcal{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathcal{E}_{u,v} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

for all central singularities $g$ we consider.

We now focus on the two kinds of families $G_\mu$ addressed in Theorem 2.2.2. Here we use the results of [BGV03, Appendix A] regarding the $\mathbb{Z}_q$-tangent space $T(g)$, for various $\mathbb{Z}_q$-equivariant germs $g$.

1. If $q \geq 5$ and $K_0(0,0) \neq 0$, then

$$T(g) = \mathcal{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathcal{E}_{u,v} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

cf. [BGV03, Appendix A]. Therefore, a complement of $T(g)$ is the 2-dimensional real vector space $V$ defined by

$$V = \mathbb{R} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix} \right\}.$$

The projection $\Pi : \mathcal{E}^2_{u,v} \to V$ is given by

$$\Pi(M,N) = \big(\,\mathrm{Re}(M(0,0)), \mathrm{Im}(M(0,0))\,\big),$$

since, for $M \in \mathcal{E}_{u,v}$:

$$M(u,v) = \mathrm{Re}(M(0,0)) + i\,\mathrm{Im}(M(0,0)) \quad \mathrm{mod}\,\mathcal{M}.$$

Therefore, (2.23) holds if $n \geq 2$, and there are two parameters, say, $\mu_1$ and $\mu_2$, such that

$$\det\left(\frac{\partial(\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0)))}{\partial(\mu_1,\mu_2)}\right)\bigg|_{\mu=0} \neq 0.$$

Equivalently, (2.23) holds if the map

$$\mu \mapsto (\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0)))$$

is a submersion at $\mu = 0$.

2. If $q \geq 7$ and $K_0(0,0) = 0$, but $D_u K_0(0,0) \neq 0$, then

$$T(g) = \mathcal{M}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathcal{E}_{u,v} \begin{pmatrix} v \\ 0 \end{pmatrix} \oplus \mathcal{E}_{u,v} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

cf. [BGV03, Appendix A]. Therefore, a complement of $T(g)$ is the 4-dimensional real vector space $V$ defined by

$$V = \mathbb{R}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix}, \begin{pmatrix} iu \\ 0 \end{pmatrix} \right\}.$$

In this case, the projection $\Pi : \mathcal{E}^2_{u,v} \to V$ is given by

$$\Pi(M,N) = \big(\,\mathrm{Re}(M(0,0)), \mathrm{Im}(M(0,0)), \mathrm{Re}(D_u M(0,0)), \mathrm{Im}(D_u M(0,0))\,\big),$$

since, for $M \in \mathcal{E}_{u,v}$:

$$\begin{aligned} M(u,v) = \mathrm{Re}(M(0,0)) + i\,\mathrm{Im}(M(0,0)) + \mathrm{Re}(D_u M(0,0))\,u \\ + i\,\mathrm{Im}(D_u M(0,0))\,u \qquad \mathrm{mod}(\mathcal{M}^2 + \langle v \rangle). \end{aligned}$$

Therefore, (2.23) holds if $n \geq 4$, and there are four parameters, say, $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$, such that

$$\det\left(\frac{\partial(\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0)), \mathrm{Re}(D_u K_\mu(0,0)), \mathrm{Im}(D_u K_\mu(0,0)))}{\partial(\mu_1,\mu_2,\mu_3,\mu_4)}\right)\bigg|_{\mu=0} \neq 0.$$

Equivalently, (2.23) holds if the map

$$\mu \mapsto (\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0)), \mathrm{Re}(D_u K_\mu(0,0)), \mathrm{Im}(D_u K_\mu(0,0)))$$

is a submersion at $\mu = 0$.

This concludes the proof of Theorem 2.2.2. □

Although we do not address the cases $q = 3$ and $q = 4$ in our leading examples, the versality of unfoldings can also be established in these cases, based on the expressions for the $\mathbb{Z}_q$-tangent space of the central singularity derived for these cases in [BGV03]. The precise conditions, solving the recognition problem, are summarized in the following result.

**Lemma 2.4.1.**

1. *If $q = 3$, the family $G_\mu$ is a versal unfolding of $g$ if the map*

$$\mu \mapsto (\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0))) \tag{2.24}$$

   *is a submersion at $\mu = 0$, irrespective of $K_\mu$, i.e., irrespective of whether $D_u K_0(0,0)$ is non-zero or not.*

2. *If $q = 4$, the family $G_\mu$ is a versal unfolding of $g$ if $|D_u K_0(0,0)/L_0(0,0)| \neq 0, 1$, if $n \geq 3$, and if the map*

$$\mu \mapsto (\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0)), \mathrm{Re}(D_u K_\mu(0,0))) \tag{2.25}$$

   *is a submersion at $\mu = 0$.*

*Proof.*

1. If $q = 3$, we have $T(g) = \mathcal{M}\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathcal{E}_{u,v}\begin{pmatrix}0\\1\end{pmatrix}$, irrespective of $K_\mu$, that is, irrespective of whether $D_u K_0(0,0)$ is non-zero or not. In this case the family $G_\mu(z)$ is a generic unfolding of $g$ if

$$\det\left(\frac{\partial(\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0)))}{\partial(\mu_1, \mu_2)}\right)\bigg|_{\mu=0} \neq 0,$$

   which is equivalent to (2.24).

2. The case $q = 4$ is interesting in the sense that a *modulus parameter* appears: The $\mathbb{Z}_q$-tangent space is equal to

$$T(g) = \mathcal{M}^2\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathcal{M}\begin{pmatrix}0\\1\end{pmatrix} \oplus \mathbb{R}\left\{\begin{pmatrix}v\\0\end{pmatrix}, \begin{pmatrix}iv\\0\end{pmatrix}, \begin{pmatrix}0\\i\end{pmatrix}, \begin{pmatrix}au\\1\end{pmatrix}, \begin{pmatrix}iau\\0\end{pmatrix}\right\},$$

   with $a = D_u K_0(0,0)/L_0(0,0)$. If $|a| \neq 0, 1$ a complement of $T(g)$ is the three-dimensional real vector space $V$ defined by

$$V = \mathbb{R}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}i\\0\end{pmatrix}, \begin{pmatrix}u\\0\end{pmatrix}\right\},$$

since, for $M \in \mathcal{E}_{u,v}$:

$$M(u,v) = \mathrm{Re}(M(0,0)) + i\,\mathrm{Im}(M(0,0)) + \mathrm{Re}(D_u M(0,0))\,u \quad \mathrm{mod}\, T(g).$$

Therefore, (2.23) holds if $n \geq 3$, and there are three parameters, say, $\mu_1$, $\mu_2$ and $\mu_3$, such that

$$\det\left(\frac{\partial(\mathrm{Re}(K_\mu(0,0)), \mathrm{Im}(K_\mu(0,0)), \mathrm{Re}(D_u K_\mu(0,0)))}{\partial(\mu_1, \mu_2, \mu_3)}\right)\Bigg|_{\mu=0} \neq 0,$$

which is equivalent to (2.25).

□

Lemma 2.4.1 yields explicit expressions for universal unfoldings of $\mathbb{Z}_q$-equivariant germs $g$ of the form

$$g(z) = z\,k(|z|^2) + \ell\,\overline{z}^{q-1} + O(|z|^q),$$

with $k(0) = 0$ and $\ell \neq 0$, also for the cases $q = 3$ and $q = 4$. These are of the form

$$\begin{aligned} G_\mu(z) &= g(z) + (\mu_1 + i\mu_2)\,z, && \text{if } q = 3; \\ G_\mu(z) &= g(z) + (\mu_1 + i\mu_2)\,z + \mu_3\,z\,|z|^2, && \text{if } q = 4 \text{ and } |D_u k(0)/\ell| \neq 0, 1. \end{aligned}$$

**Normal forms for $\mathbb{Z}_q$-equivariant germs.** In [BGV03] we determined normal forms for generic and mildly degenerate $\mathbb{Z}_q$-equivariant germs of functions from $\mathbb{C}$ to $\mathbb{C}$. These normal forms are low-degree polynomials in $z$ and $\overline{z}$. For completeness, we recall these results here.

The $\mathbb{Z}_q$*-tangent space constant theorem* states that, for $g \in \mathcal{E}(\mathbb{Z}_q)$, the germ $g+tp$ is equivalent to $g$ for all $p \in \mathcal{E}(\mathbb{Z}_q)$ if

$$T(g + tp) = T(g),$$

for $t \in [0, 1]$. Using this theorem, in [BGV03] we obtained the following classification of the simplest $\mathbb{Z}_q$-germs

$$g(z) = K(u,v)z + L(u,v)\overline{z}^{q-1},$$

in case $K(0,0) = 0$, and $L(0,0) \neq 0$:

1. If $D_u K(0,0) \neq 0$, then a normal form of the germ $g$ is:

$$\begin{cases} \overline{z}^2, & \text{if } q = 3; \\ a\,z\,|z|^2 + \overline{z}^3, & \text{if } q = 4, \text{ with } a = |D_u K(0,0)/L(0,0)| \neq 0, 1; \\ z\,|z|^2 + \overline{z}^{q-1}, & \text{if } q \geq 5. \end{cases}$$

   The parameter $a$ in the case $q = 4$ is a modulus: The germs corresponding to different values of this modulus have the same codimension. We refer to [BGV03] for further details, and for a derivation of this fact.

2. If $D_u K(0,0) = 0$ and $D_u^2 K(0,0) \neq 0$, then a normal form of $g$ is:

$$z\,|z|^4 + \overline{z}^{q-1}, \qquad \text{if } q \geq 7.$$

## 2.5 Conclusion and future work

The reduction algorithm, and the solution to the recognition problem presented in this chapter allow for a classification of the resonance sets of non-degenerate and mildly degenerate families of planar diffeomorphisms a fixed point of which undergoes a Hopf-Neĭmark-Sacker bifurcation with its characteristic array of resonance sets organizing the alternation of periodic and quasi-periodic dynamics. With this approach the shape of the boundary of the resonance set can be determined up to a diffeomorphism. More precisely, we are able to determine a normal form for the family. The resonance sets appearing in normal forms of generic families undergoing a non-degenerate resonant HNS bifurcation are easy to analyze, see Figure 2.2. In the mildly degenerate case the boundary of the resonance set is a more complicated algebraic subset of 4-dimensional space, which is analyzed in Chapter 3.

To determine the boundary of the resonance set of the actual family, and not just of its normal form, our algorithm needs to be extended with a module computing the parametrization involved in the transformation that brings the Lyapunov-Schmidt reduced family into normal form under $\mathbb{Z}_q$-equivariant contact equivalence. Such an extension would bring our current work under the same paradigm as [BHLV03], where fine-tuned Gröbner basis methods are applied to compute both the normal form and the transformation bringing the system at hand into this normal form. The current context, in which all germs are $\mathbb{Z}_q$-equivariant, is more complicated, so further work needs to be done in order to fully solve the recognition problem.